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How the treatment or conversion changes the sense of mathematical objects

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Summary. *In this paper I will demonstrate a consequence at times manifest in the semiotic transformations involving the treatment and conversion of a semiotic representation whose sense derives from a shared practice. The shift from one representation of a mathematical object to another via transformations, on the one hand maintains the meaning of the object itself, but on the other can change its sense. This is demonstrated in detail through a specific example, while at the same time it is collocated within a broad theoretical framework that poses fundamental questions concerning mathematical objects, their meanings and their representations.*

1. Preliminary remarks

It often happens, at any school level, in mathematical situations that can also be very different between each other, that we are surprised by a statement that suddenly reveals a missed conceptual construction regarding topics that instead appeared thoroughly acquired.

We will give a roundup of examples that we found in the past years and we will try to give one of the possible explanations of this phenomenon, analysing in particular an example.

We will refer to Radford (2004) (Fig. 1) where one can find this diagram that we appreciate because of its attempt to put in the right place the ideas of *sense* and *understanding*.
Note how the sense allows to give different "presentations" of the same object, whereas the understanding allows to say that the synthesis of these presentations leads to the understanding of the object.

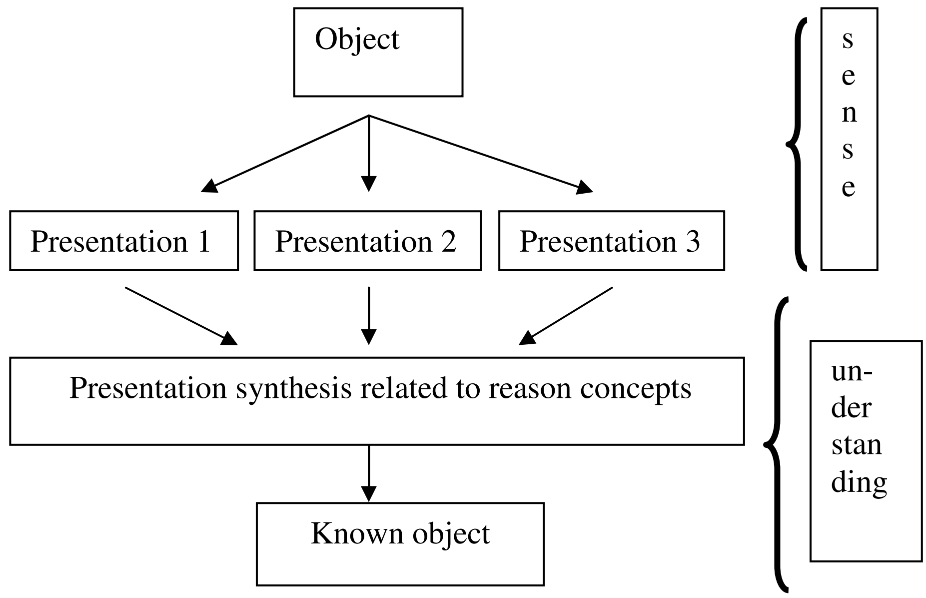

Fig. 1

2. Mathematical object, its shared meaning and its semiotic representations: the narration of an episode

In a fifth class (pupils aging 10 years) of an Italian Primary School, the teacher has conducted an introductory lesson in a-didactic situation concerning the first elements of probability, in which the pupils construct, with at least the use of some examples, the idea of "event" and “the probability of simple events”. As an example, the teacher uses a normal die with six faces to study the random results from a statistical point of view. From this emerges a frequency probability which is, however, interpreted in the classical sense. The teacher then proposes the following exercise:

Calculate the probability of the following event: the result of an even number when throwing the die.

Pupils discuss in groups and above all sharing strategies devised under the direction of the teacher decide that the answer is expressed by the fraction $\frac{3}{6}$ because «the possible results are 6 (at the denominator) while the results that render the event true are 3 (at the numerator)».

After having institutionalised the construction of this knowledge, satisfied by the result of the experience and the fact that the outcome has been rapidly obtained and the pupils have shown considerable skill in handling fractions, the teacher proposes that, on the basis of the equivalence between $\frac{3}{6}$ and $\frac{50}{100}$, it is also possible to express the probability by writing 50% and that this is indeed more expressive, since it means that the probability of such a result is a half, in terms of the generality of all possible events which is 100. A pupil observes that «so we can also use the [fraction] $\frac{1}{2}$», and the proposal is verified through the explanation of the pupil, rapidly accepted by all and once again institutionalised by the teacher.

If we analyse the different semiotic representations of the same event which emerge during this activity – “the result of throwing a die is an even number” – it is possible to identify at least the following:

- semiotic register: natural language: probability that the result of throwing a die is an even number
- semiotic register: the language of fractions: $\frac{3}{6}$, $\frac{50}{100}$, $\frac{1}{2}$
- semiotic register: the language of percentages: 50%.

Each of the preceding semiotic representations is the signifier which follows from a preceding single meaning (Duval, 2003). The shared “sense” of what was being developed together was always the same and therefore the mathematical practice carried out and described led to semiotic transformations for which the final results were easily accepted:

- conversion: from the semiotic representation expressed in the natural language register to the written form $\frac{3}{6}$
- treatment: from the written forms $\frac{3}{6}$ and $\frac{1}{2}$ to $\frac{50}{100}$
- conversion: from the written form $\frac{50}{100}$ to 50%.

At the end of the sequence the pupils are asked if the fraction $\frac{4}{8}$ can be used to represent the same event, since it is equivalent to $\frac{3}{6}$. *The answer is negative, unanimous and without hesitation*. Even the teacher, who had previously handled the situation with confidence, asserts that «$\frac{4}{8}$ *cannot* represent the event because a die has 6 faces and not 8». Pressed to consider further the question, the teacher adds: «There are not only dice with 6 faces, but also dice with 8 faces. In that case, yes, the fraction $\frac{4}{8}$ can represent the result of throwing a die is an even number».

3. A symbolism for semiotic principles

In other studies we have already used the following symbols (D'Amore, 2001, 2003a,b, and elsewhere):
Hereafter we will use:

r^m $=_{df}$ m^{th} semiotic register
$R^m_i(A)$ $=_{df}$ i^{th} semiotic representation of concept A in the semiotic register r^m

(m = 1, 2, 3, …; i = 1, 2, 3, …).
The following diagram illustrates the question even more clearly (with reference to Duval, 1993):

characteristics of the semiotic: *representation – treatment – conversion* [imply different cognitive activities] (m, n, i, j, h = 1, 2, 3, …):

concept A to be represented → choice of distinctive features of A

REPRESENTATION of A [$R^{m}{}_{i}(A)$] in a given semiotic register r^{m}

transformation of representation *TREATMENT*

transformation of register *CONVERSION*

new representation (i≠j) [$R^{m}{}_{j}(A)$] in the *same* semiotic register r^{m}

new representation (h≠i, h≠j) [$R^{n}{}_{h}(A)$] in a *different* semiotic register r^{n} (n≠m).

4. Let's turn back to the episode

- There exists a mathematical object (meaning) O_1 to represent: the probability that the result of throwing a die is an even number;
- a *sense* is ascribed to the object on the basis of a presumable shared experience which is part of a social practice shared in the class;

- a semiotic register r^m is chosen in order to represent O_1: $R^m{}_i(O_1)$;
- a treatment is effected: $R^m{}_i(O_1) \to R^m{}_j(O_1)$;
- a conversion is effected: $R^m{}_i(O_1) \to R^n{}_h(O_1)$;
- $R^m{}_j(O_1)$ is interpreted and the mathematical object (meaning) O_2 is recognised in it;
- $R^n{}_h(O_1)$ is interpreted and the mathematical object (meaning) O_3 is recognised in it.

What is the relationship between O_2, O_3 and O_1?
Identity can be recognised; and this means that there is a previous knowledge, on the basis of which identity itself can be pointed out.
But we can avoid to recognise identity, so the "interpretation" is or seems different, and in this case we lose the *sense* of the original starting-object (meaning) O_1.
Duval too treats the question of different representation of the same object (Duval 2006).

5. Conclusion

What we would like to emphasize here is how the sense of a mathematical object is more complex than is considered within the usual pair (object and its representations). There are semantic links between pairs of this kind:

(object, its representation) – (object, its *other* representation)

These links are due to semiotic transformations between the representations of the same object, but then cause the loss of sense of the initial object. Although both object and semiotic transformations are the result of shared practices, the outcomes of the transformations can require *other* attributions of sense through *other* shared practices. This is highly suggestive for all studies of ontology and knowledge.
The phenomenon described can be used to complete the picture proposed by Duval of the role of the multiple representations of an object in understanding it and also to break the vicious circle of the paradox. Every representation carries with it a *different* "subsystem of practices", from which emerge *different* objects (previously called O_i, $i \geq 1$). But the articulation of these objects within a more general system requires a change of perspective, a movement into another context in

which the search for a *common structure* is a part of the system of global practices in which distinct "partial objects" play a role.
The progressive development of the use of different representations undoubtedly enriches the meaning, the knowledge and the understanding of the object, but also its complexity. In one sense the mathematical object presents itself as unique, in another as multiple.
What is then the nature of the mathematical object? The only reply would seem to be "structural, formal, grammatical" (in the epistemological sense) together with "global, mental, structural" (in the psychological sense) which we as subjects construct within our brains as our experience is progressively enriched.

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